

LINKS BETWEEN ORTHOGONAL ARRAYS, ASSOCIATION SCHEMES AND PARTIAL GEOMETRIC DESIGNS

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ABSTRACT. In this paper, we show how certain three-class association schemes and orthogonal arrays give rise to partial geometric designs. We also investigate the connections between partial geometric designs and certain regular graphs having three or four distinct eigenvalues, three-class association schemes, orthogonal arrays of strength two and particular linear codes. We give various characterizations of these graphs, association schemes and orthogonal arrays in terms of partial geometric designs. We also give a list of infinite families of directed strongly regular graphs arising from the partial geometric designs obtained in this paper.

1. INTRODUCTION

Partial geometric designs (also known as $1\frac{1}{2}$ -designs) were recently shown to produce directed strongly regular graphs [12]. In [26] and [25] we uncovered which difference sets and difference families produce partial geometric designs. Here we take the next step and explore the link between these designs and other combinatorial structures. Specifically, we establish connections with strongly regular graphs, certain wreath product association schemes, three-class association schemes, and specific orthogonal arrays of strength two.

It is well-known that many strongly regular graphs give rise to symmetric 2-designs and partial geometric designs (cf. [9], [24], [28]). In particular, every complete multipartite regular graph gives rise to a partial geometric design. Additionally, any strongly regular graph satisfying $\lambda = \mu$ gives rise to a symmetric $2-(v, k, \lambda)$ design, which is, in turn, a partial geometric design. (See, for example, [24, 27, 17].) In fact it is shown that a strongly regular graph with parameters (v, k, λ, μ) gives rise to a partial geometric design if and only if it satisfies either $k = \mu$ or $\lambda = \mu$ (cf. Section 3 below).

Every strongly regular graph is realized as a relation graph of some association scheme. In particular, a nontrivial strongly regular graph and its complement are the relation graphs of a two-class association scheme. However, there are graphs that give rise to partial geometric designs but are not realized as relation graphs of association schemes. Our investigation into finding the source of partial geometric designs begins with studying the characteristics of the graphs that give rise to such designs. We observe that some of these graphs arise as the relation graphs of certain three-class association schemes. This observation leads us to explore the links between partial geometric designs, graphs, and association schemes. Some of these association schemes come from certain orthogonal arrays of strength two and linear codes. As a consequence, we are able to find an infinite family of partial geometric designs and give a list of directed strongly regular graphs arising from these partial geometric designs.

The organization of the paper is as follows. In the following section, we introduce notation that will be used throughout and recall some basic terms from the theory of designs and association schemes.

In Section 3, we characterize the strongly regular graphs that give rise to partial geometric designs.

In Section 4, we recall that the wreath product of an arbitrary association scheme with the trivial association scheme possesses a relation graph isomorphic to a strongly regular graph. Hence such a wreath product association scheme gives rise to a partial geometric design. Conversely, if an imprimitive association scheme of class three or more contains exactly one strongly regular relation graph, then such a scheme must be isomorphic to the wreath product of a scheme with a one-class association scheme.

In Section 5, we describe parameter sets of certain three-class association schemes that give rise to partial geometric designs. In particular, we show that if a 3-class symmetric self-dual association scheme of order $3m^2$ satisfies certain parametric conditions, then its adjacency matrices A_0, A_1, A_2, A_3 satisfy the following identities for some constants α_i and β_i :

$$\begin{aligned} A_1^3 &= \beta_1 A_1 + \alpha_1 (J - A_1), \\ A_2^3 &= \beta_2 A_2 + \alpha_2 (J - A_2), \\ (A_3 + A_0)^3 &= \beta_3 (A_3 + A_0) + \alpha_3 (J - A_3 - A_0). \end{aligned}$$

In Section 6, we then provide concrete examples of such association schemes coming from Hamming codes and certain orthogonal arrays of strength two.

In Section 7, we provide the parameter sets of directed strongly regular graphs obtained from the partial geometric designs constructed in this paper by applying the relationship between partial geometric designs and directed strongly regular graphs given by Brouwer-Olmez-Song in [12]. Finally, we close with some last remarks on our construction of partial geometric designs.

2. PRELIMINARIES

Here we recall some basic facts about block designs and association schemes. We also set the notation that will be used throughout the paper.

2.1. Designs. A *block design* is a pair (P, \mathcal{B}) where P is a finite set, the elements of which are called *points*, and \mathcal{B} is a finite collection (possibly multiset) of nonempty subsets of P called *blocks*.

A *tactical configuration*, often also called a *1-design*, with parameters (v, b, k, r) is a design (P, \mathcal{B}) with $|P| = v$ and $|\mathcal{B}| = b$ such that each block consists of k points and each point belongs to r blocks. A 2 -(v, k, λ) *design* is a 1-design satisfying the added condition that every pair of distinct points is contained in exactly λ blocks.

A *partial geometric design* with parameters $(v, b, k, r; \alpha, \beta)$ is a 1-design (P, \mathcal{B}) with parameters (v, b, k, r) satisfying the ‘partial geometric’ property: For every point $x \in P$ and every block $B \in \mathcal{B}$, the number of incident point-block pairs (y, C) such that $y \in B$ and $x \in C$ is α if $x \notin B$ and is β if $x \in B$ for some constants α and β . That is,

$$|\{(y, C) : y \in B \cap C, C \ni x\}| = \begin{cases} \alpha & \text{if } x \notin B, \\ \beta & \text{if } x \in B. \end{cases}$$

If N is the point-block incidence matrix of a $(v, b, k, r; \alpha, \beta)$ -partial geometric design, then it satisfies

$$(1) \quad JN = kJ, \quad NJ = rJ, \quad NN^T N = \beta N + \alpha(J - N),$$

where N^T denotes the transpose of N , and J is the all-ones matrix. A 2 -(v, k, λ) design is partial geometric with $\alpha = k\lambda$ and $\beta = \lambda \frac{v-1}{k-1} + k\lambda - \lambda$. We will say that a partial geometric design (P, \mathcal{B}) is *symmetric* whenever $v = b$ (and so, $k = r$). When the design is symmetric, its parameters are simply denoted by $(v, k; \alpha, \beta)$, in short.

In this paper, by the phrase, “*graph $\Gamma = (V, E)$ gives rise to design (P, \mathcal{B})* ,” we mean that the adjacency matrix A of Γ is *equivalent* to the incidence matrix N of (P, \mathcal{B}) . That is, for each $v \in V$, if we let $N_v = \{x \in V : (x, v) \in E\}$ and $\mathcal{N} = \{N_v : v \in V\}$, the pair (V, \mathcal{N}) forms a design that is *isomorphic* to (P, \mathcal{B}) .¹

2.2. Association schemes and their Bose-Mesner algebras. Let X be an n -element set, and let R_0, R_1, \dots, R_d be subsets of $X \times X := \{(x, y) : x, y \in X\}$ with $R_0 = \{(x, x) : x \in X\}$. Let A_i be the $n \times n$ $\{0, 1\}$ -matrix representing R_i : i.e.,

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

The pair $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called a d -class (*symmetric*) *association scheme* if A_0, A_1, \dots, A_d satisfy the following:

- (1) $A_0 + A_1 + \dots + A_d = J$, where J is the all-ones matrix and $A_0 = I$, the identity matrix,
- (2) for each $i \in \{0, 1, \dots, d\}$, $A_i^T = A_i$,
- (3) for any $h, i, j \in \{0, 1, \dots, d\}$, there exists a constant p_{ij}^h such that

$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h.$$

The matrices A_0, A_1, \dots, A_d defined above are called the *adjacency matrices* of \mathcal{X} , and the graphs $(X, R_1), (X, R_2), \dots, (X, R_d)$, are called the *relation graphs* of \mathcal{X} . The constants p_{ij}^h are called the *intersection numbers* of \mathcal{X} , and for any $(x, y) \in R_h$

$$p_{ij}^h = |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|.$$

Let B_i , $i \in \{0, 1, \dots, d\}$, be the i th *intersection matrix* defined by

$$(B_i)_{jh} = p_{ij}^h.$$

Then $B_i B_j = \sum_{h=0}^d p_{ij}^h B_h$.

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme with its adjacency matrices A_0, A_1, \dots, A_d and intersection matrices B_0, B_1, \dots, B_d . Then the \mathbb{C} -space with basis $\{A_0, A_1, \dots, A_d\}$ is an algebra over the complex numbers, called the *Bose-Mesner algebra* of \mathcal{X} , denoted by $\mathcal{A}(\mathcal{X})$ or $\langle A_0, A_1, \dots, A_d \rangle$. The \mathbb{C} -algebra generated by $\{B_0, B_1, \dots, B_d\}$ is called the *intersection algebra* of \mathcal{X} . The Bose-Mesner algebra $\mathcal{A}(\mathcal{X})$ and the intersection algebra $\langle B_0, B_1, \dots, B_d \rangle$ are isomorphic \mathbb{C} -algebras induced by the correspondence $A_i \mapsto B_i$. (For more information, see for example, [2, 10].)

¹There exist bijections $f: V \rightarrow P$ and $\phi: \mathcal{N} \rightarrow \mathcal{B}$ such that $x \in N_v$ if and only if $f(x) \in \phi(N_v)$.

3. STRONGLY REGULAR GRAPHS WITH EITHER $k = \mu$ OR $\lambda = \mu$

Strongly regular graphs arise from various combinatorial structures, especially in connection with designs and codes. For a complete characterization of partial geometric designs as well as a thorough investigation of their connection to partial geometries and strongly regular graphs, we refer the readers to Bose, Shrikhande and Singh [7] and Neumaier [24]. In this section, we characterize which strongly regular graphs give rise to symmetric partial geometric designs.

Lemma 1. *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) . Let A be the adjacency matrix of Γ . Then $A^3 = \beta A + \alpha(J - A)$ for some integers α and β if and only if either $\lambda = \mu$ or $k = \mu$. (In this case, $\alpha = (\lambda - \mu)\mu + \mu k$ and $\beta = (\lambda - \mu)^2 + k - \mu + (\lambda - \mu)\mu + \mu k$.)*

Proof. Given a strongly regular graph Γ with parameters (v, k, λ, μ) , the adjacency matrix A of Γ satisfies the identity:

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Thus, we have that

$$A^3 = \{(\lambda - \mu)^2 + (\lambda - \mu)\mu + k\mu + k - \mu\}A + (\lambda - \mu)(k - \mu)I + \{(\lambda - \mu)\mu + \mu k\}(J - A).$$

Therefore, there exist α and β such that $A^3 = \beta A + \alpha(J - A)$ if and only if

$$(\lambda - \mu)(k - \mu) = 0, \quad (\lambda - \mu)^2 + k - \mu + (\lambda - \mu)\mu + \mu k = \beta \quad \text{and} \quad (\lambda - \mu)\mu + \mu k = \alpha.$$

Hence the proof follows. \square

Every complete multipartite strongly regular graph can be viewed as the complement of c -copies of the complete graph K_n on n vertices for some integers c and n (where $c, n \geq 2$). We denote such a graph by $\overline{cK_n}$.

Corollary 2. *The complete multipartite strongly regular graph $\overline{cK_n}$ gives rise to a symmetric partial geometric design with parameters $(cn, (c - 1)n; (c^2 - 3c + 2)n^2, (c^2 - 3c + 3)n^2)$.*

Proof. This strongly regular graph has parameters

$$(v, k, \lambda, \mu) = (cn, (c - 1)n, (c - 2)n, (c - 1)n).$$

The result now follows from Lemma 1. \square

A strongly regular graph with parameters (v, k, λ, λ) is sometimes called a (v, k, λ) -graph. The adjacency matrix A of a (v, k, λ) -graph satisfies identity $A^2 = kI + \lambda(J - I)$; therefore, it gives a symmetric 2- (v, k, λ) -design. Since a symmetric 2- (v, k, λ) -design is a partial geometric design with parameters $(v, k; k\lambda, k\lambda + k - \lambda)$, so we also have:

Corollary 3. *A (v, k, λ) -graph gives rise to a partial geometric design with parameters*

$$(v, k; k\lambda, k\lambda + k - \lambda).$$

Remark 4. (1) *We note that both the Hamming graph $H(2, 4)$ and the Shrikhande graph are $(16, 6, 2)$ -graphs. Although these two graphs are non-isomorphic they give rise to the same 2- $(16, 6, 2)$ -design. (cf. [14, Ch.2 and Ch.4].) Hence, we have a partial geometric design with parameters $(v, k; \alpha, \beta) = (16, 6; 12, 16)$ as all 2-designs are partial geometric.*

- (2) We also note that there are many (v, k, λ) -graphs: Examples of small graphs include $(35, 15, 6)$, $(35, 18, 9)$, $(36, 21, 12)$, $(45, 12, 3)$, $(63, 32, 16)$ and $(64, 36, 20)$. To see the current list of such strongly regular graphs, visit the homepage of E. Spence [27] or A. Brouwer [8].

4. WREATH PRODUCT OF A SCHEME BY A ONE-CLASS ASSOCIATION SCHEME

In this section, we establish a connection between partial geometric designs and wreath products of association schemes. We show that every wreath product association scheme in which one factor is a trivial association scheme gives rise to a partial geometric design. It follows from the fact that such a wreath product association scheme has a relation graph which is strongly regular with $k = \mu$.

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ and $\mathcal{Y} = (Y, \{S_j\}_{0 \leq j \leq e})$ be association schemes of order $|X| = m$ and $|Y| = n$, respectively. Let $\{A_i\}_{0 \leq i \leq d}$ and $\{C_j\}_{0 \leq j \leq e}$ be the sets of adjacency matrices of \mathcal{X} and \mathcal{Y} , respectively. Then the adjacency matrices of the wreath product $\mathcal{X} \wr \mathcal{Y}$ of \mathcal{X} and \mathcal{Y} are

$$I_n \otimes A_0, I_n \otimes A_1, \dots, I_n \otimes A_d, C_1 \otimes J_m, C_2 \otimes J_m, \dots, C_e \otimes J_m,$$

where $A \otimes C = (a_{ij}C)$ denotes the Kronecker product of $A = (a_{ij})$ and C . With this ordering of the adjacency matrices, the relation matrix of $\mathcal{X} \wr \mathcal{Y}$ is given by

$$R(\mathcal{X} \wr \mathcal{Y}) = I_n \otimes R(\mathcal{X}) + [R(\mathcal{Y}) + d(J_n - I_n)] \otimes J_m,$$

where $R(\mathcal{X}) = \sum_{h=0}^d hA_h$ and $R(\mathcal{Y}) = \sum_{h=0}^e hC_h$.

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme with its Bose-Mesner algebra $\mathcal{A}(\mathcal{X}) = \langle A_0, A_1, \dots, A_d \rangle$. For any relations R_i and R_j , define

$$R_i R_j := \{R_h : p_{ij}^h \neq 0\}.$$

Then, for a nonempty subset H of $\{0, 1, \dots, d\}$, $\{R_h\}_{h \in H}$ is called a *closed subset* if $R_i R_j \subseteq \{R_h\}_{h \in H}$ for any $i, j \in H$. If $\{R_h\}_{h \in H}$ is a closed subset, then the \mathbb{C} -space with basis $\{A_h\}_{h \in H}$ is a subalgebra of $\mathcal{A}(\mathcal{X})$, called a *Bose-Mesner subalgebra* of \mathcal{X} , denoted by \mathcal{A}_H or $\mathcal{A}_H(\mathcal{X})$.

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme with $\mathcal{A}(\mathcal{X}) = \langle A_0, A_1, \dots, A_d \rangle$, and let $\{R_h\}_{h \in H}$ be a closed subset of \mathcal{X} . Let $\mathcal{Y} = (Y, \{S_j\}_{0 \leq j \leq e})$ be an association scheme with its Bose-Mesner algebra $\mathcal{A}(\mathcal{Y}) = \langle C_0, C_1, \dots, C_e \rangle$. Let $\{S_g\}_{g \in G}$ be a closed subset of \mathcal{Y} . We say that the Bose-Mesner subalgebras $\mathcal{A}_H(\mathcal{X})$ and $\mathcal{A}_G(\mathcal{Y})$ are *exactly isomorphic* if there is a bijection $\pi : H \rightarrow G$ such that the linear map from $\mathcal{A}_H(\mathcal{X})$ to $\mathcal{A}_G(\mathcal{Y})$ induced by $A_h \mapsto C_{\pi(h)}$ for $h \in H$ is an algebra isomorphism.

The following properties of the Bose-Mesner algebra of a wreath product of association schemes (See [3, 4]) are useful for our discussion.

Lemma 5. *Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme. Let \mathcal{K}_n denote the one-class association scheme whose nontrivial relation graph is the complete graph K_n . If $\mathcal{X} = \mathcal{Y} \wr \mathcal{K}_n$ for an association scheme $\mathcal{Y} = (Y, \{S_j\}_{0 \leq j \leq e})$ and \mathcal{K}_n , then $e = d - 1$ and, by renumbering R_1, R_2, \dots, R_d if necessary, the following hold.*

(i) *$\{R_0, R_1, \dots, R_{d-1}\}$ is a closed subset of \mathcal{X} such that the Bose-Mesner subalgebra $\mathcal{A}_{[d-1]}$ with basis $\{A_0, A_1, \dots, A_{d-1}\}$ is exactly isomorphic to the Bose-Mesner algebra of \mathcal{Y} .*

(ii) Let k_0, k_1, \dots, k_d denote the valency of \mathcal{X} , and let $m = \sum_{i=0}^{d-1} k_i$. Then $|Y| = m$ and

$$(2) \quad A_i A_d = k_i A_d, \quad 1 \leq i < d;$$

$$(3) \quad A_d^2 = m(n-1)(J_{mn} - A_d) + m(n-2)A_d$$

and

$$(4) \quad A_d^3 = m^2(n^2 - 3n + 2)(J_{mn} - A_d) + m^2(n^2 - 3n + 3)A_d.$$

Proof. Let C_j be the adjacency matrix of \mathcal{Y} corresponding to S_j , $0 \leq j \leq e$. Then the adjacency matrices A_i of $\mathcal{Y} \wr \mathcal{K}_n$ can be expressed as follows:

$$A_0 = I_n \otimes C_0, \quad A_1 = I_n \otimes C_1, \dots, A_{d-1} = I_n \otimes C_e, \quad A_d = (J_n - I_n) \otimes J_m.$$

Thus, $e = d - 1$ and

$$\begin{aligned} A_d^2 &= ((J_n - I_n) \otimes J_m)^2 &= (J_n - I_n)^2 \otimes J_m^2 \\ &= ((n-2)J_n + I_n) \otimes mJ_m \\ &= m(n-2)(J_n - I_n) \otimes J_m + m(n-1)I_n \otimes J_m \\ &= m(n-2)A_d + m(n-1)(J_{mn} - A_d). \end{aligned}$$

This verifies that (3) holds. We then obtain the identity (4) by multiplying both sides of (3) by A_d and using (3) again to derive the desired form. It is clear that the valency k_i of R_i is equal to that of S_i , for $1 \leq i \leq d-1$, and (2) holds. Obviously, $\{R_1, R_2, \dots, R_{d-1}\}$ is a closed subset, and the Bose-Mesner subalgebra with basis $\{A_0, A_1, \dots, A_{d-1}\}$ is exactly isomorphic to the Bose-Mesner algebra of \mathcal{Y} . \square

From (4), we have the following.

Theorem 6. *Let $\mathcal{X} = \mathcal{Y} \wr \mathcal{K}_n$ be the wreath product of association schemes \mathcal{Y} and \mathcal{K}_n . Let the adjacency matrices of \mathcal{X} be ordered such a way that $A_d = (J_n - I_n) \otimes J_m$ where $m = |Y|$. Then the A_d can be viewed as the incidence matrix of a symmetric partial geometric design with parameters $(mn, m(n-1); m^2(n^2 - 3n + 2), m^2(n^2 - 3n + 3))$.*

Proof. It immediately follows from the fact that the relation graph (X, R_d) is a multipartite strongly regular graph. \square

5. CERTAIN THREE-CLASS SELF-DUAL ASSOCIATION SCHEMES OF ORDER $3m^2$

In this section we show that the relation graphs of certain three-class association schemes give rise to partial geometric designs.

In order to represent the parameters of an association scheme in a compact form, we recall the definition of its character table. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a symmetric association scheme of order $|X| = n$ with adjacency matrices A_0, A_1, \dots, A_d . Let $E_0 = \frac{1}{n}J, E_1, \dots, E_d$ denote the primitive idempotents in $\mathcal{A}(\mathcal{X})$. Then there are $p_j(i), q_i(j) \in \mathbb{C}$ for all $i, j \in \{0, 1, \dots, d\}$ such that

$$A_j = \sum_{i=0}^d p_j(i)E_i \quad \text{and} \quad E_i = \frac{1}{n} \sum_{j=0}^d q_i(j)A_j.$$

The $(d+1) \times (d+1)$ matrices P and Q whose (i, j) -entries are defined by

$$P_{ij} = p_j(i) \quad \text{and} \quad Q_{ij} = q_j(i)$$

are called the *1st eigenmatrix* and *2nd eigenmatrix* of \mathcal{X} , respectively. The first eigenmatrix is often called the *character table* of the association scheme. We note that $PQ = nI$. An association scheme is said to be (*formally*) *self-dual* if $P = Q$. Next, note that if none of the relation graphs of a symmetric three-class association scheme are strongly regular, then every relation graph has four distinct eigenvalues [28].

Lemma 7. *Let \mathcal{Z} be a three-class symmetric association scheme of order $3m^2$ for some positive integer $m \equiv 0 \pmod{3}$. Then the following two statements are equivalent.*

(1) *The character table P of \mathcal{Z} is given by*

$$P = \begin{bmatrix} 1 & m(m-1) & m(m+1) & (m-1)(m+1) \\ 1 & m & 0 & -m-1 \\ 1 & 0 & -m & m-1 \\ 1 & -m & m & -1 \end{bmatrix}.$$

(2) *\mathcal{Z} is self-dual and its adjacency matrices A_i satisfy the following identities:*

$$A_1^3 = m^2 A_1 + \frac{1}{3} m^2 (m-1)(m-2) J$$

$$A_2^3 = m^2 A_2 + \frac{1}{3} m^2 (m+1)(m+2) J$$

$$(A_3 + I)^3 = m^2 (A_3 + I) + \frac{1}{3} m^2 (m-1)(m+1) J.$$

Proof. First, it is straightforward to verify that $P^2 = 3m^2 I$, and so \mathcal{Z} is self-dual. We can also calculate all the intersection numbers of \mathcal{Z} directly from the character table by using the basic identity:

$$p_{ij}^h = \frac{1}{n \cdot k_h} \sum_{\nu=0}^3 p_i(\nu) p_j(\nu) p_h(\nu) k_\nu$$

for $h, i, j \in \{0, 1, 2, 3\}$. Namely, the intersection matrices are given by:

$$\begin{aligned} B_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ m(m-1) & \frac{1}{3}m(m-2) & \frac{1}{3}m(m-1) & \frac{1}{3}m^2 - m \\ 0 & \frac{1}{3}m(m+1) & \frac{1}{3}m(m-1) & \frac{1}{3}m^2 \\ 0 & \frac{1}{3}m(m-2) - 1 & \frac{1}{3}m(m-1) & \frac{1}{3}m^2 \end{bmatrix} \\ B_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3}m(m+1) & \frac{1}{3}m(m-1) & \frac{1}{3}m^2 \\ m(m+1) & \frac{1}{3}m(m+1) & \frac{1}{3}m(m+2) & \frac{1}{3}m^2 + m \\ 0 & \frac{1}{3}m(m+1) & \frac{1}{3}m(m+2) - 1 & \frac{1}{3}m^2 \end{bmatrix} \\ B_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{3}m(m-2) - 1 & \frac{1}{3}m(m-1) & \frac{1}{3}m^2 \\ 0 & \frac{1}{3}m(m+1) & \frac{1}{3}m(m+2) - 1 & \frac{1}{3}m^2 \\ (m-1)(m+1) & \frac{1}{3}m(m+1) & \frac{1}{3}m(m-1) & \frac{1}{3}m^2 - 2 \end{bmatrix}. \end{aligned}$$

Second, by applying the basic identity

$$B_i B_j = \sum_{h=0}^3 p_{ij}^h B_h,$$

we obtain

$$B_i^3 = B_i^2 B_i = p_{ii}^0 B_0 B_i + p_{ii}^1 B_1 B_i + p_{ii}^2 B_2 B_i + p_{ii}^3 B_3 B_i$$

for $i = 1, 2, 3$, in the intersection algebra of \mathcal{Z} . Note that $B_0 B_i = B_i$. By plugging the values of p_{ij}^h in the second identity and using the first identity repeatedly, we obtain the following identities:

$$B_1^3 = m^2 B_1 + \frac{1}{3} m^2 (m-1)(m-2)(B_0 + B_1 + B_2 + B_3)$$

$$B_2^3 = m^2 B_2 + \frac{1}{3} m^2 (m+1)(m+2)(B_0 + B_1 + B_2 + B_3)$$

$$(B_3 + B_0)^3 = m^2 (B_3 + B_0) + \frac{1}{3} m^2 (m-1)(m+1)(B_0 + B_1 + B_2 + B_3)$$

Finally, we see that the desired identities are deduced from these identities by the isomorphism between the Bose-Mesner algebra $\langle A_0, A_1, A_2, A_3 \rangle$ and intersection algebra $\langle B_0, B_1, B_2, B_3 \rangle$. Thus (1) implies (2).

Conversely, if we multiply both sides of each identity in (2) by the all-ones vector \mathbf{j} , then for instance, from the first identity, we have

$$A_1^3 \mathbf{j} = m^2 A_1 \mathbf{j} + \frac{1}{3} m^2 (m-1)(m-2) J \mathbf{j}$$

or equivalently,

$$k_1^3 \mathbf{j} = m^2 k_1 \mathbf{j} + m^4 (m-1)(m-2) \mathbf{j}.$$

That is, we have

$$k_1^3 = m^2 k_1 + m^4 (m-1)(m-2);$$

and so, $k_1 = m(m-1)$. Similarly, we find $k_2 = m(m+1)$ and $k_3 + 1 = m^2$. Furthermore, since the all-ones matrix J has only one non-zero eigenvalue (which is $3m^2$) and the rest of them are zeros, from the identity $A_1^3 - m^2 A_1 = \frac{1}{3} m^2 (m-1)(m-2) J$, we see that the remaining possible eigenvalues for A_1 are $0, m, -m$. This is also true for the cases of A_2 and $A_3 + I$. Also we know that by the self-duality of the association scheme, possible multiplicities for these eigenvalues are k_1, k_2 , and k_3 . Now by using the row- and column-orthogonality of the character table, we can arrange the eigenvalues of A_1, A_2 and A_3 to obtain P . This completes the proof. \square

As an immediate consequence of this lemma, we have the following.

Theorem 8. *Let \mathcal{Z} be a three-class association scheme, and let A_0, A_1, A_2, A_3 be its adjacency matrices. Suppose that the character table P of \mathcal{Z} is given by*

$$P = \begin{bmatrix} 1 & m(m-1) & m(m+1) & (m-1)(m+1) \\ 1 & m & 0 & -m-1 \\ 1 & 0 & -m & m-1 \\ 1 & -m & m & -1 \end{bmatrix}.$$

Then \mathcal{Z} gives rise to three symmetric partial geometric designs coming from the incidence matrices A_1, A_2 and $A_3 + A_0$. In this case, the parameters $(v, k; \alpha, \beta)$ of corresponding partial geometric designs are given by

$$\begin{aligned} & (3m^2, m(m-1); \frac{1}{3}m^2(m^2-3m+2), \frac{1}{3}m^2(m^2-3m+5)), \\ & (3m^2, m(m+1); \frac{1}{3}m^2(m^2+3m+2), \frac{1}{3}m^2(m^2+3m+5)), \\ & (3m^2, m^2; \frac{1}{3}m^2(m^2-1), \frac{1}{3}m^2(m^2+2)). \end{aligned}$$

Remark 9. Having a character table of an association scheme is equivalent to having the intersection matrices since the character table essentially generates all intersection numbers and vice versa. However, it is possible for more than one association scheme to have the same character table, and thus, the same intersection numbers. For example, it is well-known that the two strongly regular graphs with the same parameters $(16, 6, 2, 2)$ discussed in Remark 4 in Section 3, are relation graphs of two distinct two-class association schemes.

6. THE ASSOCIATION SCHEMES FROM CODES AND ORTHOGONAL ARRAYS

In this section, we give concrete examples of the three-class association schemes described in the previous section. Our examples come as fusion schemes of the Hamming association scheme $H(d, 3)$ and are also obtained from a family of linear orthogonal arrays of strength two. As a result, we see an interesting link between the three-class association schemes and linear orthogonal arrays.

6.1. From Hamming schemes. Here we show that a three-class fusion scheme of $H(d, 3)$ for each odd $d \geq 3$ gives rise to partial geometric designs.

First we recall the definition of the Hamming scheme $H(d, q)$. Let S be a q -element set and let

$$V := S^d = \{(x_1, x_2, \dots, x_d) : x_j \in S, j = 1, 2, \dots, d\}.$$

Define the association relation between any $\mathbf{x} = (x_1, x_2, \dots, x_d)$ and $\mathbf{y} = (y_1, y_2, \dots, y_d) \in V$ according to the Hamming distance

$$\delta(\mathbf{x}, \mathbf{y}) := |\{j \in \{1, 2, \dots, d\} : x_j \neq y_j\}|;$$

that is, define

$$(\mathbf{x}, \mathbf{y}) \in R_i \Leftrightarrow \delta(\mathbf{x}, \mathbf{y}) = i.$$

Then $(V, \{R_i\}_{0 \leq i \leq d})$ is an association scheme called the d -class *Hamming scheme*, over S , denoted by $H(d, q)$.

First, we show that among the Hamming schemes of class three, $H(3, 3)$ is the only Hamming scheme that gives rise to partial geometric designs.

Proposition 10. *The relation graphs of the Hamming scheme $H(3, q)$ give rise to partial geometric designs if and only if $q = 3$.*

Proof. The eigenmatrices and the first intersection matrix of $H(3, q)$ are given by

$$P = Q = \begin{bmatrix} 1 & 3(q-1) & 3(q-1)^2 & (q-1)^3 \\ 1 & 2q-3 & (q-1)(q-3) & -(q-1)^2 \\ 1 & q-3 & -2q+3 & q-1 \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3(q-1) & q-2 & 2 & 0 \\ 0 & 2(q-1) & 2(q-2) & 3 \\ 0 & 0 & q-1 & 3(q-2) \end{bmatrix}$$

By direct calculation, it is shown that the intersection matrices satisfy the identity:

$$B_1^3 = 3(q^2 - 3q + 2)I + (q^2 + 3q - 3)B_1 + 6(q-2)B_2 + 6B_3.$$

By the algebra isomorphism between the Bose-Mesner algebra and the intersection algebra, it then follows that

$$A_1^3 = 3(q^2 - 3q + 2)I + (q^2 + 3q - 3)A_1 + 6(q-2)A_2 + 6A_3.$$

Therefore, when $q = 3$, we have

$$A_1^3 = 15A_1 + 6(J - A_1).$$

However, for $H(3, q)$ with $q \neq 3$, there is no way that we can express A_1^3 as a linear combination of A_1 and $J - A_1$. Therefore, we see that the first relation graph of $H(3, q)$ gives rise to a partial geometric design if and only if $q = 3$. By a similar calculation, we can verify that when $q = 3$,

$$B_2^3 = 69B_2 + 60(B_0 + B_1 + B_3), \quad (B_3 + B_0)^3 = 33(B_3 + B_0) + 24(B_1 + B_2),$$

or equivalently,

$$A_2^3 = 69A_2 + 60(J - A_2), \quad (A_3 + I)^3 = 33(A_3 + I) + 24(J - A_3 - I).$$

This completes the proof. \square

We note that $H(3, 3)$ has the same parameters as the three-class association scheme described in Lemma 7 with $m = 3$. Although there is no other Hamming scheme whose relation graphs give rise to partial geometric designs, there exists a three-class fusion scheme of $H(d, 3)$, for each odd $d \geq 3$, whose relation graphs give rise to partial geometric designs. Kageyama, Saha and Das in [21, Theorem 2] introduced the following three-class fusion scheme \mathcal{F} of $H(d, 3)$ which will be called the *KSD-scheme* in what follows.

Theorem 11. [21] *Consider Hamming scheme $H(d, 3) = (V, \{R_i\}_{0 \leq i \leq d})$ with $d = 2l + 1$ for $l \geq 1$, and let $S_0 = R_0$ and*

$$S_j = \bigcup_{i=0}^{[(d-j)/3]} R_{3i+j}, \text{ for } j = 1, 2, 3$$

where $[(d-j)/3]$ denotes the greatest integer less than or equal to $(d-j)/3$. Then $\mathcal{F} = (V, \{S_0, S_1, S_2, S_3\})$ is a three-class association scheme with the following intersection matrices:

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3^{2l} + (-1)^l 3^l & 3^{2l-1} + 2(-1)^l 3^{l-1} & 3^{2l-1} + (-1)^l 3^{l-1} & 3^{2l-1} + (-1)^l 3^l \\ 0 & 3^{2l-1} - (-1)^l 3^{l-1} & 3^{2l-1} + (-1)^l 3^{l-1} & 3^{2l-1} \\ 0 & 3^{2l-1} + 2(-1)^l 3^{l-1} - 1 & 3^{2l-1} + (-1)^l 3^{l-1} & 3^{2l-1} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 3^{2l-1} - (-1)^l 3^{l-1} & 3^{2l-1} + (-1)^l 3^{l-1} & 3^{2l-1} \\ 3^{2l} - (-1)^l 3^l & 3^{2l-1} - (-1)^l 3^{l-1} & 3^{2l-1} - 2(-1)^l 3^{l-1} & 3^{2l-1} - (-1)^l 3^l \\ 0 & 3^{2l-1} - (-1)^l 3^{l-1} & 3^{2l-1} - 2(-1)^l 3^{l-1} - 1 & 3^{2l-1} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 3^{2l-1} + 2(-1)^l 3^{l-1} - 1 & 3^{2l-1} + (-1)^l 3^{l-1} & 3^{2l-1} \\ 0 & 3^{2l-1} - (-1)^l 3^{l-1} & 3^{2l-1} - 2(-1)^l 3^{l-1} - 1 & 3^{2l-1} \\ 3^{2l} - 1 & 3^{2l-1} - (-1)^l 3^{l-1} & 3^{2l-1} + (-1)^l 3^{l-1} & 3^{2l-1} - 2 \end{bmatrix}.$$

Remark 12. For every d , the association scheme \mathcal{F} belongs to the family of association schemes \mathcal{Z} described in Lemma 7. In fact, if l is odd, the two schemes \mathcal{F} and \mathcal{Z} have the same parameters with $m = 3^l$. For each even integer l , the parameters of \mathcal{F} are the same as those of \mathcal{Z} with $m = 3^l$ and the first and second association relations switched.

Corollary 13. For each $l \geq 1$, the relation graphs of the association scheme \mathcal{F} above give rise to three non-isomorphic symmetric partial geometric designs with parameters

$$\begin{aligned} & (3^{2l+1}, 3^{2l} + (-1)^l 3^l; 3^{4l-1} + (-1)^l 3^{3l} + 2 \cdot 3^{2l-1}, 3^{4l-1} + (-1)^l 3^{3l} + 5 \cdot 3^{2l-1}), \\ & (3^{2l+1}, 3^{2l} - (-1)^l 3^l; 3^{4l-1} - (-1)^l 3^{3l} + 2 \cdot 3^{2l-1}, 3^{4l-1} - (-1)^l 3^{3l} + 5 \cdot 3^{2l-1}), \\ & (3^{2l+1}, 3^{2l}; 3^{4l-1} - 3^{2l-1}, 3^{4l-1} + 2 \cdot 3^{2l-1}). \end{aligned}$$

Proof. The proof directly follows from the following identities of the intersection matrices:

$$\begin{aligned} B_1^3 &= (3^{4l-1} + (-1)^l 3^{3l} + 5 \cdot 3^{2l-1}) B_1 + (3^{4l-1} + (-1)^l 3^{3l} + 2 \cdot 3^{2l-1}) (B_0 + B_2 + B_3), \\ B_2^3 &= (3^{4l-1} - (-1)^l 3^{3l} + 5 \cdot 3^{2l-1}) B_2 + (3^{4l-1} - (-1)^l 3^{3l} + 2 \cdot 3^{2l-1}) (B_0 + B_1 + B_3), \\ (B_3 + B_0)^3 &= (3^{4l-1} + 2 \cdot 3^{2l-1}) (B_3 + B_0) + (3^{4l-1} - 3^{2l-1}) (B_1 + B_2). \end{aligned}$$

Thus three designs whose incidence matrices are A_1 , A_2 and $A_3 + I$ are obtained from this association scheme. \square

6.2. From orthogonal arrays of strength two. Our search for three-class association schemes whose relation graphs give rise to partial geometric designs continues in the context of orthogonal arrays and linear codes. Here we find another way to construct the KSD-schemes using orthogonal arrays of strength 2 coming from suitable linear codes. Taking the runs (codewords) as the elements of the underlying set and defining association relations according to the Hamming distances between the codewords, we obtain three-class association schemes that are isomorphic to KSD-schemes.

Let S be a set of q -symbols where $q \geq 2$, and let $I := \{1, 2, \dots, m\}$. Let $X = S^I$ be the set of all maps from I to S . Note that we can view each element $\mathbf{x} \in X$ as an m -tuple (x_1, x_2, \dots, x_m) with symbols x_i in S . A *code* is simply a subset C of X . In the case where $S = \mathbb{F}_q$ and C forms a vector space over \mathbb{F}_q , we call C a *linear code*. Next, an N -element subset Y of X , viewed as an $N \times m$ array of symbols, is called an *orthogonal array of strength t* and *index λ* if every $N \times t$ subarray contains all possible q^t t -tuples exactly λ times. Following the notation of [20], we denote an orthogonal array Y with the above parameters by $\text{OA}(N, m, q, t)$ where $\lambda = N/q^t$. The rows of an $\text{OA}(N, m, q, t)$ are sometimes called the *runs* of the orthogonal array. In what follows, the runs of Y will be denoted $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N$ with $\mathbf{y}_i = (y_{i1}, y_{i2}, \dots, y_{im})$. It will be clear whether we consider Y as an N -set or as an array from the context.

An orthogonal array is *linear* if it takes a finite field as its symbol set, and its rows form a vector space over the field. We will show that for every positive integer l , there exists a linear orthogonal array $\text{OA}(3^{2l+1}, 2l+3, 3, 2)$ coming from a certain linear $[2l+3, 2l+1]_3$ -code which gives an association scheme isomorphic to a KSD-scheme. First we have the following example.

Example 14. Two mutually orthogonal Latin cubes of order 3 give an orthogonal array $OA(27, 5, 3, 2)$. The transpose of the orthogonal array is expressed as the following 5×27 array M .

$$M = \begin{pmatrix} 000 & 000 & 000 & 111 & 111 & 111 & 222 & 222 & 222 \\ 000 & 111 & 222 & 000 & 111 & 222 & 000 & 111 & 222 \\ 012 & 012 & 012 & 012 & 012 & 012 & 012 & 012 & 012 \\ 000 & 111 & 222 & 222 & 000 & 111 & 111 & 222 & 000 \\ 012 & 120 & 201 & 012 & 120 & 201 & 012 & 120 & 201 \end{pmatrix}$$

A 3-class association scheme is then obtained as follows:

(i) Let $Y := \{\mathbf{y}_i : i = 1, 2, \dots, 27\}$ be the set of columns of M (runs of $OA(27, 5, 3, 2)$).

(ii) Let $R_0 = \{(\mathbf{y}_i, \mathbf{y}_i) : \mathbf{y}_i \in Y\}$, and let

$$\begin{aligned} R_1 &= \{(\mathbf{y}_i, \mathbf{y}_j) \in Y \times Y : \delta(\mathbf{y}_i, \mathbf{y}_j) = 2 \text{ or } 5\} \\ R_2 &= \{(\mathbf{y}_i, \mathbf{y}_j) \in Y \times Y : \delta(\mathbf{y}_i, \mathbf{y}_j) = 3\} \\ R_3 &= \{(\mathbf{y}_i, \mathbf{y}_j) \in Y \times Y : \delta(\mathbf{y}_i, \mathbf{y}_j) = 4\} \end{aligned}$$

Then $\mathcal{Y} = (Y, \{R_i\}_{0 \leq i \leq 3})$ is an association scheme. Its character table is given by

$$P = Q = \begin{bmatrix} 1 & 6 & 8 & 12 \\ 1 & 3 & -4 & 0 \\ 1 & -3 & -1 & 3 \\ 1 & 0 & 2 & -3 \end{bmatrix},$$

and we have the following identities in the Bose-Mesner algebra of \mathcal{Y} :

$$A_1^3 = 15A_1 + 6(J - A_1), \quad (A_2 + I)^3 = 33(A_2 + I) + 24(J - A_2 - I) \quad \text{and} \quad A_3^3 = 69A_3 + 60(J - A_3).$$

Remark 15. (1) The above 3-class association scheme is shown to be isomorphic to $H(3, 3)$ discussed in Proposition 10. Here the isomorphism is established by the fact that all Hamming schemes except for $H(2, 4)$ are uniquely determined by their intersection numbers.

(2) This $OA(27, 5, 3, 2)$ can be also obtained as the codewords generated by the three vectors $[1, 0, 0, 1, 1]$, $[0, 1, 0, 0, 1]$ and $[0, 0, 1, 1, 0]$ over \mathbb{F}_3 in the five dimensional Hamming space $H(5, 3)$.

In the rest of this section, our alphabet S (the symbol set) will be finite field \mathbb{F}_q of order q . We will denote the code of length n and size N over the alphabet \mathbb{F}_q by $(n, N)_q$ or by $(n, N, d)_q$ if the minimum distance d is known. If the code is an m -dimensional subspace of the n -dimensional vector space \mathbb{F}_q^n , we denote it by $[n, m]_q$ -code (or $[n, m]$ -code if the field is clear from the context). We denote the dual code of a $[n, m]$ -code \mathcal{C} , by \mathcal{C}^\perp and its minimum distance by d^\perp , the dual distance of \mathcal{C} . Two linear codes are *isomorphic* if one can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a nonzero element of the field. Two linear orthogonal arrays are considered to be the same if the associated codes are isomorphic as linear codes.

In order to describe how we find linear orthogonal arrays $OA(3^{2l+1}, 2l+3, 3, 2)$ from $[2l+3, 2l+1]_3$ -codes, we recall a few useful facts which link orthogonal arrays and linear codes. R. C. Bose [5] explicitly specified how the strength of a linear orthogonal array is determined by the associated code. Ph. Delsarte [16] specified connections between the codes and orthogonal arrays. Our results are based on the following theorem which states a special case of a more profound result due to him. It suffices for our construction of

orthogonal arrays as we chiefly concentrate on the linear case. (See [20, Ch. 4] for more information.)

Theorem 16. [16] *If \mathcal{C} is a $(n, N, d)_q$ linear code over \mathbb{F}_q with dual distance d^\perp then the codewords of \mathcal{C} form the rows of an $OA(N, n, q, d^\perp - 1)$ with entries from \mathbb{F}_q . Conversely, the rows of a linear $OA(N, n, q, t)$ over \mathbb{F}_q form a $(n, N, d)_q$ linear code over \mathbb{F}_q with dual distance $d^\perp \geq t + 1$. If the orthogonal array has strength t but not $t + 1$, d^\perp is precisely $t + 1$.*

Remark 17. *In particular, given q, m and fixed strength t , if the $m \times n$ generator matrix G for any $[n, m]_q$ -code \mathcal{C} has the property that every t -columns of G are linearly independent vectors in \mathbb{F}_q^m over \mathbb{F}_q , then the q^m codewords of \mathcal{C} form a linear orthogonal array $OA(q^m, n, q, t)$. For this, we note that G generates q^m codewords all of which become the runs of the $q^m \times n$ orthogonal array, say M . The $q^m \times t$ subarray obtained by taking any t -columns of M contains linear combinations of the rows of the corresponding t -columns of G ; and it contains each of q^t t -tuples of symbols exactly q^{m-t} times.*

Thus, in order to obtain orthogonal arrays of strength 2, we simply look at the generator matrices of all $[n, m]_q$ -codes over \mathbb{F}_q and pick the ones whose dual distance is 3. By Theorem 16 and Remark 17, we know that if such a $[2l + 3, 2l + 1]_3$ -code yields an orthogonal array, then it must be a $OA(3^{2l+1}, 2l + 3, 3, 2)$. Now we demonstrate an infinite family of orthogonal arrays that give rise to the three-class association schemes we seek.

Consider the $[2l + 3, 2l + 1]_3$ -code \mathcal{C} with generator matrix

$$G = \left[\begin{array}{c|cc} & 1 & 1 \\ & 0 & 1 \\ & 1 & 0 \\ & 0 & 0 \\ & \vdots & \vdots \\ & 0 & 0 \\ I_{2l+1} & & \end{array} \right]$$

where I_{2l+1} denotes the $(2l + 1) \times (2l + 1)$ identity matrix. Then the dual code \mathcal{C}^\perp is generated by

$$G^\perp = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$$

with its weight distribution $(1, 0, 0, 4, 2, 2, 0, \dots, 0)$. Thus, by Theorem 16, the codewords of the $(2l + 3, 3^{2l+1})_3$ -code \mathcal{C} with dual distance 3, form the rows of an $OA(3^{2l+1}, 2l + 3, 3, 2)$.

Next, consider the $[2l + 3, 2l + 1]_3$ -codes \mathcal{C}' with generator matrix

$$G' = \left[\begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ & \vdots & \vdots \\ & 0 & 0 \\ I_{2l+1} & & \end{array} \right].$$

We can identify \mathcal{C}' as the Hamming space $H(2l + 1, 3)$ by viewing \mathcal{C}' as a natural embedding of \mathbb{F}_3^{2l+1} in \mathbb{F}_3^{2l+3} . Moreover, suppose we define

$$M_0 = \{\mathbf{0}\}, \quad M_i = \{\mathbf{x} \in \mathcal{C} \setminus \{\mathbf{0}\} : \delta_H(\mathbf{0}, \mathbf{x}) \equiv i \pmod{3}\} \quad \text{for } i = 1, 2, 3,$$

and the sets M'_i for \mathcal{C}' in the same manner. Then $\{M_i : i = 0, 1, 2, 3\}$ forms a partition of \mathcal{C} . In the same spirit, $\{M'_i : i = 0, 1, 2, 3\}$ forms a partition of \mathcal{C}' according to their Hamming weights. By Theorem 11, we know that

$$\{|M'_i| : i = 0, 1, 2, 3\} = \{1, 3^l(3^l + 1), 3^l(3^l - 1), 3^{2l} - 1\}.$$

By establishing a vector space isomorphism between \mathcal{C}' and \mathcal{C} as below, we can also see that

$$\{|M_i| : i = 0, 1, 2, 3\} = \{1, 3^l(3^l + 1), 3^l(3^l - 1), 3^{2l} - 1\}.$$

For this, let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2l+1}$ denote the rows of G' , as the basis vectors for \mathcal{C}' , and $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2l+1}$ denote the rows of G , which form a basis for \mathcal{C} . Define a map $\phi : \mathcal{C}' \rightarrow \mathcal{C}$ by

$$\phi(\mathbf{e}_1) = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3, \quad \phi(\mathbf{e}_2) = \mathbf{r}_2, \quad \phi(\mathbf{e}_3) = \mathbf{r}_3,$$

and for $i > 3$,

$$\phi(\mathbf{e}_i) = \begin{cases} \mathbf{r}_i + \mathbf{r}_{i+1}, & \text{if } i \equiv 0 \pmod{2} \\ \mathbf{r}_{i-1} + 2\mathbf{r}_i, & \text{if } i \equiv 1 \pmod{2} \end{cases}.$$

It is clear that ϕ is a vector space isomorphism. Furthermore, this map ϕ maps M'_i to M_i setwise; namely, $\phi(M'_1) = M_2$, $\phi(M'_2) = M_1$ and $\phi(M'_3) = M_3$. For this, we recall that the sets M'_i and M_i were defined according to the Hamming weight of the codewords. Notice that a weight- s codeword \mathbf{x} can be expressed as $\sum_{h=1}^s \alpha_h \mathbf{e}_{j_h}$ for some $\alpha_h \in \mathbb{F}_3^*$ and some s -set $\{j_1, j_2, \dots, j_s\}$ with $1 \leq j_1 < j_2 < \dots < j_s \leq 2l+1$ (where $\mathbb{F}_3^* = \mathbb{F}_3 - \{0\}$). So, we can express

$$M'_i = \left\{ \sum_{h=1}^s \alpha_h \mathbf{e}_{j_h} : 1 \leq j_1 < j_2 < \dots < j_s \leq 2l+1, \alpha_h \in \mathbb{F}_3^*, s \equiv i \pmod{3}, 1 \leq s \leq 2l+1 \right\}.$$

Then, by direct computation, it can be verified that

$$\phi(M'_i) = \left\{ \sum_{h=1}^s \alpha_h \phi(\mathbf{e}_{j_h}) : \sum_{h=1}^s \alpha_h \mathbf{e}_{j_h} \in M'_i \right\} \subseteq \begin{cases} M_2 & \text{if } i = 1 \\ M_1 & \text{if } i = 2 \\ M_3 & \text{if } i = 3 \end{cases}.$$

So, it follows that

$$\phi(M'_1) = M_2, \quad \phi(M'_2) = M_1, \quad \phi(M'_3) = M_3$$

as $|M'_1| + |M'_2| + |M'_3| = |M_1| + |M_2| + |M_3| = 3^{2l+1} - 1$.

As a consequence, we have the following.

Theorem 18. (1) For each $l \in \mathbb{N}$, there exists a linear $[2l+3, 2l+1]_3$ -code \mathcal{C} with dual distance 3, such that \mathcal{C} is partitioned into M_0, M_1, M_2, M_3 where $M_0 = \{\mathbf{0}\}$ and for $i = 1, 2, 3$, $M_i = \{\mathbf{x} \in \mathcal{C} \setminus \{\mathbf{0}\} : \delta_H(\mathbf{0}, \mathbf{x}) \equiv i \pmod{3}\}$ with cardinalities $|M_i| \in \{3^{2l} - 3^l, 3^{2l} + 3^l, 3^{2l} - 1\}$. In this case, any linear $[2l+3, 2l+1]_3$ -code equivalent to \mathcal{C} is an orthogonal array $OA(3^{2l+1}, 2l+3, 3, 2)$.

(2) *Defining relations on \mathcal{C} by*

$$\begin{aligned} R_0 &= \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathcal{C}\} \\ R_1 &= \{(\mathbf{x}, \mathbf{y}) \in \mathcal{C} \times \mathcal{C} : \delta_H(\mathbf{x}, \mathbf{y}) \equiv 1(\text{mod } 3)\} \\ R_2 &= \{(\mathbf{x}, \mathbf{y}) \in \mathcal{C} \times \mathcal{C} : \delta_H(\mathbf{x}, \mathbf{y}) \equiv 2(\text{mod } 3)\} \\ R_3 &= \{(\mathbf{x}, \mathbf{y}) \in \mathcal{C} \times \mathcal{C} : \mathbf{x} \neq \mathbf{y}, \delta_H(\mathbf{x}, \mathbf{y}) \equiv 0(\text{mod } 3)\} \end{aligned}$$

we obtain a three-class association scheme $\mathcal{W} = (\mathcal{C}, \{R_i\}_{0 \leq i \leq 3})$ which has the same parameters as those for the KSD-scheme \mathcal{F} for $d = 2l + 1$ defined in Theorem 11.

Proof. The statement (1) is summary of what we have discussed earlier. For statement (2), we recall that, under the map ϕ , all codewords of weight i in \mathcal{C}' are mapped to the codewords of weight $2i \pmod{3}$ in \mathcal{C} ; and thus, for any $\mathbf{x}, \mathbf{y} \in \mathcal{C}'$, $\delta_H(\mathbf{x}, \mathbf{y}) \equiv i \pmod{3}$ if and only if $\delta_H(\phi(\mathbf{x}), \phi(\mathbf{y})) \equiv 2i \pmod{3}$. Therefore, the three-class association scheme \mathcal{W} defined on \mathcal{C} and the KSD-scheme \mathcal{F} defined on $H(2l + 1, 3)$ in Theorem 11 share the same parameter sets. That is, if the parameters of KSD scheme are p_{ij}^h , then those for \mathcal{W} are $p_{\sigma(i)\sigma(j)}^{\sigma(h)}$ where $\sigma = (12)$ is the transposition in S_3 . \square

We now give an example to illustrate what we have discussed in this subsection.

Example 19. *The linear $[7, 5]_3$ -code \mathcal{C} generated by*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

has weight distribution $(1, 4, 8, 24, 60, 82, 56, 8)$ and dual weight distribution $(1, 0, 0, 4, 2, 2, 0, 0)$. This code \mathcal{C} (and any code equivalent to \mathcal{C}) gives $OA(243, 7, 3, 2)$. By defining association relations by

$$\begin{aligned} R_0 &= \{(x, x) \mid x \in X\} \\ R_1 &= \{(x, y) \mid \delta_H(x, y) \in \{1, 4, 7\}\} \\ R_2 &= \{(x, y) \mid \delta_H(x, y) \in \{2, 5\}\} \\ R_3 &= \{(x, y) \mid \delta_H(x, y) \in \{3, 6\}\}, \end{aligned}$$

we obtain a three-class association scheme, described as \mathcal{W} . The intersection matrices of \mathcal{W} are given by

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 72 & 21 & 24 & 18 \\ 0 & 30 & 24 & 27 \\ 0 & 20 & 24 & 27 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 30 & 24 & 27 \\ 90 & 30 & 33 & 36 \\ 0 & 30 & 32 & 27 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 20 & 24 & 27 \\ 0 & 30 & 32 & 27 \\ 80 & 30 & 24 & 25 \end{bmatrix}.$$

and for this association scheme, we have the following identities in the Bose-Mesner algebra of \mathcal{W} :

$$\begin{aligned} A_1^3 &= 1593A_1 + 1512(J - A_1), \\ A_2^3 &= 3051A_2 + 2970(J - A_2), \\ (A_3 + I)^3 &= 2241(A_3 + I) + 2160(J - A_3 - I). \end{aligned}$$

Therefore, we obtain three partial geometric designs with parameters

$$(v, k; \alpha, \beta) = (243, 72; 1512, 1593), (243, 90; 2160, 2241), (243, 81; 2970, 3051).$$

7. DIRECTED STRONGLY REGULAR GRAPHS

Given a partial geometric design, Theorems 2.1 and 2.2 in [12] tell us how to construct two directed strongly regular graphs. One is defined on the flags of the design, while the other is defined on the antiflags. Here we list the parameters of the directed strongly regular graphs that we can obtain from the partial geometric designs constructed above.

Definition 20. A directed strongly regular graph (DSRG) with parameters $(v, k, t, \lambda', \mu')$ is a directed graph on v vertices without loops such that

- (1) Every vertex has in-degree and out-degree k ,
- (2) Every vertex has t out-neighbors which are also in-neighbors, and
- (3) For any two distinct vertices x and y , the number of directed paths from x to y of length 2 is λ' if $x \rightarrow y$ and is μ' otherwise.

We observe that the adjacency matrix of a DSRG with parameters $(v, k, t, \lambda', \mu')$ has the property that

$$AJ = JA = kJ \quad \text{and} \quad A^2 = tI + \lambda'A + \mu'(J - I - A).$$

Theorem 21. [12] Let (P, \mathcal{B}) be a 1-design. The following three statements are equivalent.

- (1) (P, \mathcal{B}) is a partial geometric design.
- (2) The directed graph Γ defined by $V(\Gamma) = \{(p, B) \in P \times \mathcal{B} : p \in B\}$ with the adjacency

$$(p, B) \rightarrow (q, C) \text{ if and only if } (p, B) \neq (q, C) \text{ and } p \in C,$$

is a DSRG.

- (3) The directed graph Γ' defined by $V(\Gamma') = \{(p, B) \in P \times \mathcal{B} : p \notin B\}$ with adjacency

$$(p, B) \rightarrow (q, C) \text{ if and only if } p \in C,$$

is a DSRG.

Remark 22. According to this theorem, any symmetric partial geometric design with parameters $(v, k; \alpha, \beta)$ gives rise to two DSRGs whose parameters $(v, k, t, \lambda', \mu')$ are given by

$$(v(v - k), k(v - k), k^2 - \alpha, k^2 - \beta, k^2 - \alpha) \text{ and } (vk, k^2 - 1, \beta - 1, \beta - 2, \alpha).$$

Therefore, from the partial geometric designs we have obtained in this paper, (in Theorem 8, Corollary 13 and Theorem 18) we obtain the DSRGs with the following parameters.

TABLE 1. Parameters of DSRGs obtained from partial geometric designs

(Here $m = 3^l$ for every positive integer l .)

v	k	t	λ'	μ'
$3m^3(2m+1)$	$m^2(m-1)(2m+1)$	$\frac{1}{3}m^2(2m^2-3m+1)$	$\frac{1}{3}m^2(2m^2-3m-2)$	$\frac{1}{3}m^2(2m^2-3m+1)$
$3m^3(m-1)$	$m^2(m-1)^2-1$	$\frac{1}{3}m^2(m^2-3m+5)-1$	$\frac{1}{3}m^2(m^2-3m+5)-2$	$\frac{1}{3}m^2(m^2-3m+2)$
$3m^3(2m-1)$	$m^2(m+1)(2m-1)$	$\frac{1}{3}m^2(2m^2+3m+1)$	$\frac{1}{3}m^2(2m^2+3m-2)$	$\frac{1}{3}m^2(2m^2+3m+2)$
$3m^3(m+1)$	$m^2(m+1)^2-1$	$\frac{1}{3}m^2(m^2+3m+5)-1$	$\frac{1}{3}m^2(m^2+3m+5)-2$	$\frac{1}{3}m^2(m^2+3m+2)$
$3m^2(2m^2+1)$	$(m^2-1)(2m^2+1)$	$\frac{1}{3}(2m^2-1)(m^2-1)$	$\frac{1}{3}(2m^4-8m^2+3)$	$\frac{1}{3}(2m^2-1)(m^2-1)$
$3m^2(m^2-1)$	$(m^2-1)^2-1$	$\frac{1}{3}m^2(m^2+2)-1$	$\frac{1}{3}m^2(m^2+2)-2$	$\frac{1}{3}m^2(m^2-1)$

8. CONCLUDING REMARKS

We examined graphs, association schemes, and orthogonal arrays as possible sources of partial geometric designs. At the end of our search we focused on the orthogonal arrays associated with a particular family of three-class association schemes. However, as we mentioned in the Introduction, there are graphs that give rise to partial geometric designs but are not realized as relation graphs of association schemes. In fact, there are many such graphs obtained from orthogonal arrays. On the set of runs of an orthogonal array, by defining the adjacency of any two codewords according to their Hamming distance, we obtain many graphs that give rise to partial geometric designs. So as not to extend this paper much further, we shall close with just one example. As such examples are abundant, further research is required on the topic.

Example 23. Consider the $[7, 5]$ -code \mathcal{C} over \mathbb{F}_3 with generator matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

This code \mathcal{C} has weight distribution $(1, 0, 12, 34, 42, 96, 46, 12)$ and dual weight distribution $(1, 0, 0, 2, 0, 0, 6, 0)$. Hence it gives an $OA(3^5, 7, 3, 2)$ with $\lambda = 3^3$. As before, the code \mathcal{C} (or

any code that are equivalent to \mathcal{C}) define $X = \mathcal{C}$ and

$$\begin{aligned} R_0 &= \{(x, x) \mid x \in X\} \\ R_1 &= \{(x, y) \mid \delta_H(x, y) \in \{1, 4, 7\}\} \\ R_2 &= \{(x, y) \mid \delta_H(x, y) \in \{2, 5\}\} \\ R_3 &= \{(x, y) \mid \delta_H(x, y) \in \{3, 6\}\} \end{aligned}$$

Then the partition $\{R_i\}_{0 \leq i \leq 3}$ of $X \times X$ does not form an association scheme. However, the adjacency matrices A_i of the graphs (X, R_i) for $i = 1, 2, 3$ satisfy the following identities:

$$\begin{aligned} A_1^3 &= 1215A_1 + 486(J - A_1), \\ A_2^3 &= 5589A_2 + 4860(J - A_2), \\ (A_3 + I)^3 &= 2673(A_3 + I) + 1944(J - A_3 - I). \end{aligned}$$

Thus, taking each of A_1, A_2 and $A_3 + I$ as the incidence matrix of a symmetric design, we get three partial geometric designs with parameters

$$(v, k; \alpha, \beta) = (243, 54; 486, 1215), (243, 108; 4860, 5589), (243, 81; 1944, 2673).$$

REFERENCES

- [1] T. Atsumi, A study of orthogonal arrays from the point of view of design theory, *J. Combin. Theory*, A **35**(1982), 241–251.
- [2] E. Bannai and T. Ito, *Algebraic combinatorics. I. Association schemes*. Benjamin/Cummings, Menlo Park, CA, 1984.
- [3] G. Bhattacharyya, Terwilliger algebras of wreath products of association schemes, Ph.D. Dissertation, Iowa State University, 2008.
- [4] G. Bhattacharyya, S.-Y. Song and R. Tanaka, On Terwilliger algebras of wreath products of one-class association schemes, *J. Alg. Combin.* **31**(3) (2010), 455–466.
- [5] R. C. Bose, On some connections between the design of experiments and information theory, *Bull. Internat. Statist. Inst.*, **38**(4) (1961), 257–271.
- [6] C. R. Bose and K. A. Bush, Orthogonal arrays of strength two and three, *Ann. Math. Statist.* **23**(1952), 508–524.
- [7] R. C. Bose, S. S. Shrikhande, N. M. Singhi: Edge regular multigraphs and partial geometric designs with an application to the embedding of quasi-residual designs. *Colloquie Internazionale sulle Teorie Combinatorie* 1 (1976) 49–81.
- [8] A. Brouwer, Parameters of strongly regular graphs listed at <http://www.win.tue.nl/~aeb/graphs/srg/srgtab.html>
- [9] A. E. Brouwer, Strongly Regular Graphs, in *The CRC Handbook of Combinatorial Designs*, (C.J. Colbourn, J.H. Dinitz; eds), CRC Press, New York, (1996), 667 – 685.
- [10] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin (1989).
- [11] A. E. Brouwer, S. A. Hobart: Parameters of directed strongly regular graphs, <http://homepages.cwi.nl/~aeb/math/dsrg/dsrg.html>.
- [12] A. E. Brouwer, O. Olmez and S. Y. Song, Directed strongly regular graphs from $1\frac{1}{2}$ -designs. *Europ J Combin.* **33**(6) (2012) 1174 – 1177.
- [13] A. R. Calderbank and J.-M. Goethals, On a pair of dual subschemes of the Hamming scheme $H_n(q)$, *Europ. J. Combinatorics* **6** (1985), 133–147.
- [14] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*. Cambridge University Press, 1991.
- [15] J. Degraer and K. Coolsaet, Classification of three-class association schemes using backtracking with dynamic variable ordering, *Discrete Mathematics*, **300**(13) (2005), 71 – 81.
- [16] Ph. Delsarte, An algebraic approach to the association scheme of coding theory, *Philips Res. Reports Suppl.*, **10**(1973), Ph.D. Thesis.

- [17] J. M. Goethals and J. J. Seidel, strongly regular graphs derived from combinatorial designs, *Canad. J. Math.* **22** (1970), 597–614.
- [18] W. H. Haemers, *Eigenvalue Techniques in Design and Graph Theory*, Mathematical Centre Tracts 121, Mathematisch Centrum, Amsterdam, (1980).
- [19] W. H. Haemers and E. Spence, The Pseudo-Geometric Graphs for Generalized Quadrangles of order $(3, t)$, *European J. Combin.*, **22**(6) (2001), 839 – 845.
- [20] A. S. Hedayat, N. J. A. Sloane and J. Stufken, *Orthogonal Arrays: Theory and Applications*, (Springer series in statistics), Springer-Verlag, New York, 1999.
- [21] S. Kageyama, G. M. Saha and A. D. Das, Reduction of the number of associate classes of hypercubic association schemes, *Ann. Inst. Statist. Math.* **30** (1978), Part A, 115 – 123.
- [22] R. Mathon and E. Spence, On 2-(45,12,3) designs, *Journal of Combinatorial Designs*, **4**(3) (1996), 155 – 177.
- [23] B. D. McKay and E. Spence, The Classification of Regular Two-Graphs on 36 and 38 Vertices, *Australas. J. Combin.*, **24** (2001), 293 – 300.
- [24] A. Neumaier, $t\frac{1}{2}$ -designs, *J. Combin. Thry.* (A) **28** (1980), 226–248.
- [25] K. Nowak, O. Olmez and S. Y. Song, Partial geometric difference families, *J. Combin. Des.*, to appear.
- [26] O. Olmez, Symmetric $1\frac{1}{2}$ -designs and $1\frac{1}{2}$ -difference sets, *J Combin Des.*, **22** (2014), no. 6, 252–269.
- [27] E. Spence, Strongly Regular Graphs, URL: <http://www.maths.gla.ac.uk/~es/srgraphs.html>, (2004).
- [28] E. van Dam, Three-class association schemes. *J. Algebraic Combin.* **10** (1999), no. 1, 69–107.
- [29] Y. L. Zhang, On schematic orthogonal arrays of strength two, *Ars Combin.* **91** (2009), 147–163.

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